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A New Method for General Solution Of System Of Higher-Order Linear Differential Equations

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Abstract- *This paper presents a new method for solving system of higher-order linear differential equations (HLDEs) with constant coefficients; however the same idea can be extended to variable coefficients. Using the basic concept of inverse of matrix and variation of parameters, we develop a new method to solve system of HLDEs. Proposed method works for any right hand side function, so-called vector forcing function f (x) of given system of HLDEs. Selected examples are presented using proposed method to show the efficiency.*

Key words: Systems of higher-order linear differential equations, Variation of parameters, Inverse of matrix, General solution.

I INTRODUCTION

The systems of higher-order linear differential equations (HLDEs) have been vigorously pursued by many researchers and engineers and developed many different method to solve the system, for example, see [1-10]. Naturally, the systems of HLDEs arise in many applications of nuclear reactors, multi-body systems, vibrating wires in magnetic fields, models of electrical circuits, mechanical systems, robotic modeling, and diffusion processes etc.

We consider the following type of systems of n linear differential equations of order $m > 0$,

$$A_m \frac{d^m}{dx^m} u(x) + \cdots + A_1 \frac{d}{dx} u(x) + A_0 u(x) = f(x), \tag{1}$$

where, for $i = 0, \ldots, n$, A_i are coefficient matrices of order $n \times n$, $f(x) = (f_1(x), \ldots, f_n(x))^T$ and $u(x) = (u_1(x), \ldots, u_n(x))^T$ are an n-dimensional vector forcing function and unknown vector respectively. If the lading coefficient A_m is non-singular, then the system of HLDEs (1) is called as of *first kind*, otherwise it is called *second kind*. In this paper, we present a new method to solve the system (1) of first kind with constant coefficients; however the same idea can be extended to variable coefficients. For obtaining the general solution of given system, we use the basic concepts of inverse of matrix and variation of parameters formula.

Rest of the paper is organized as follows: In Section II, we present the proposed method to solve system of HLDEs and selected examples are presented in Section III to show the efficiency of proposed method.

II A NEW METHOD FOR SYSTEM OF HIGHER-ORDER LINEAR DIFFERENTIAL EQUATIONS

Recall the system of HLDEs given in Section I

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$$L(u(x)) = A_m \frac{d^m}{dx^m} u(x) + \cdots + A_1 \frac{d}{dx} u(x) + A_0 u(x) = f(x), \tag{2}$$

where $L = A_m \frac{d^m}{dx^m} + \cdots + A_1 \frac{d}{dx} + A_0$, is $n \times n$ matrix differential operator. Now the following theorem presents the algorithm to compute the general solution of system (2), with given set of fundamental system of determinant of matrix differential operator *L*.

Theorem 1: Given a fundamental system $\{v_1(x), \ldots, v_{mn}(x)\}$, of $T = \det(L)$, then the system of higher-order LDEs

$$A_m \frac{d^m}{dx^m} u(x) + \cdots + A_1 \frac{d}{dx} u(x) + A_0 u(x) = f(x), \tag{3}$$

has following solution

$$u(x) = \begin{pmatrix} \sum_{i=1}^{n} (-1)^{i+1} \det(L_i^1) \sum_{j=1}^{mn} v_j(x) \int \frac{\det(W_j) f_i(x)}{\det(W)} dx \\ \vdots \\ \sum_{i=1}^{n} (-1)^{i+n} \det(L_i^n) \sum_{j=1}^{mn} v_j(x) \int \frac{\det(W_j) f_i(x)}{\det(W)} dx \end{pmatrix} \tag{4}$$

where *W* is the Wronskian matrix of $\{v_1(x), \ldots, v_{mn}(x)\}$ and W_i obtained from *W* by replacing the *i*-th column by *mn*-th unit vector; and $\det(L_i^k)$ denotes the determinant of *L* after removing *i*-row and *k*-th column.

Proof: Since the leading coefficient A_m of system (3) is non-singular matrix, the inverse of *L* exists with det(*L*) = *T* as scalar differential operator of order *mn*. If $\{v_1(x), \ldots, v_{mn}(x)\}$ is fundamental system of *T*. Now the solution of system (3) can be obtained as $u(x) = Adj(L) y(x)$, where $y(x) = (y_1(x), \ldots, y_n(x))^T$ is solution obtained from $Ty_i(x) = f_i(x)$ computed using the classical formulation variation of parameters as follows. The scalar differential equation $Ty(x) = f(x)$ can be reformulated as system of first order linear differential equation, say $\tilde{y}'(x) = M\tilde{y}(x) + \tilde{f}(x)$, where *M* is companion matrix. Now the solution of first order system is obtained as $\tilde{y}(x) = W \int W^{-1} \tilde{f}(x)\, dx$ and the solution $Ty(x) = f(x)$ is the first row of $\tilde{y}(x)$.

The solution (4) satisfies given system of HLDEs, as follows:

$$\begin{aligned} L(u(x)) &= L\left(\begin{pmatrix} \sum_{i=1}^{n} (-1)^{i+1} \det(L_i^1) \sum_{j=1}^{mn} v_j(x) \int \frac{\det(W_j) f_i(x)}{\det(W)} dx \\ \vdots \\ \sum_{i=1}^{n} (-1)^{i+n} \det(L_i^n) \sum_{j=1}^{mn} v_j(x) \int \frac{\det(W_j) f_i(x)}{\det(W)} dx \end{pmatrix}\right) \\ &= L\left(\begin{pmatrix} (-1)^{1+1} \det(L_1^1) & \cdots & (-1)^{1+n} \det(L_1^n) \\ \vdots & \ddots & \vdots \\ (-1)^{n+1} \det(L_n^1) & \cdots & (-1)^{n+n} \det(L_n^n) \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{mn} v_j(x) \int \frac{\det(W_j) f_1(x)}{\det(W)} dx \\ \vdots \\ \sum_{j=1}^{mn} v_j(x) \int \frac{\det(W_j) f_n(x)}{\det(W)} dx \end{pmatrix} \right) \\ &= (f_1, \ldots, f_n)^T = f \end{aligned}$$

Therefore, the solution (4) is general solution system (3).
In following section, we present selected numerical examples (system of HLDEs with constant coefficients and variable coefficients) using proposed method presented in Theorem 1.

III NUMERICAL EXAMPLES

In this section, we present couple of examples for system of linear differential equations with constant coefficients (Example 1) and variable coefficients (Example 2) respectively, to show the efficiency of proposed method in Theorem 1.

Example 1: Consider the following system of linear differential equations of order two with constant coefficients

$$\begin{aligned}&\frac{d^2}{dx^2}u_1(x)+u_1(x)-2u_2(x)=\sin(x),\\&\frac{d^2}{dx^2}u_2(x)-\frac{d}{dx}u_2(x)=e^{-3x}.\end{aligned} \tag{5}$$

The system (5) can be written in matrix notations as $L(u(x)) = f(x)$, with $L = A_2\frac{d^2}{dx^2}+A_1\frac{d}{dx}+A_0$, where

$$A_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}, A_1=\begin{pmatrix}0&0\\0&-1\end{pmatrix}, A_0=\begin{pmatrix}1&-2\\0&0\end{pmatrix} \text{ and } u(x)=\begin{pmatrix}u_1(x)\\u_2(x)\end{pmatrix}, f(x)=\begin{pmatrix}\sin(x)\\e^{-3x}\end{pmatrix}.$$

If we denote $D=\frac{d}{dx}$, then $L=\begin{pmatrix}D^2+1&-2\\0&D^2-D\end{pmatrix}$. Following procedure in Theorem 1, we have

$$Ty(x)=D^4y(x)-D^3y(x)+D^2y(x)-Dy(x)=f(x),$$

here $y(x)=(y_1(x),y_2(x))^T$. Now from Theorem 1, we have the general solution of given system (5) as follows

$$u(x)=\begin{pmatrix}u_1(x)\\u_2(x)\end{pmatrix}=\begin{pmatrix}c_1\cos(x)+c_2\sin(x)-\frac{x}{2}\cos(x)+2c_3+c_4e^x+\frac{1}{60}e^{-3x}\\c_3+c_4e^x+\frac{1}{12}e^{-3x}\end{pmatrix},$$

where c_1,c_2,c_3 and c_4 are arbitrary constants.

Example 2: Consider the following system of linear differential equations of order one with variable coefficients

$$\begin{aligned}&\frac{d}{dx}u_1(x)-x\frac{d}{dx}u_2(x)+u_1(x)-(1+x)u_2=0,\\&\frac{d}{dx}u_2(x)+u_2(x)=e^{2x}.\end{aligned} \tag{6}$$

The matrix representation of system (5) is $L(u(x))=A_1\frac{d}{dx}u(x)+A_0u(x)=f(x)$, where

$$A_1=\begin{pmatrix}1&-x\\0&1\end{pmatrix}, A_0=\begin{pmatrix}1&-1-x\\0&1\end{pmatrix} \text{ and } u(x)=\begin{pmatrix}u_1(x)\\u_2(x)\end{pmatrix}, f(x)=\begin{pmatrix}0\\e^{2x}\end{pmatrix}.$$

Matrix differential operator $L=\begin{pmatrix}D+1&-xD-1-x\\0&D+1\end{pmatrix}$, where $D=\frac{d}{dx}$. Following Theorem 1, we have the general solution of given system (6) as follows

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$$u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} = \begin{pmatrix} c_1 e^{-x} + 2c_2 x e^{-x} + c_2 e^{-x} + \frac{x}{3} e^{2x} + \frac{1}{9} e^{2x} \\ c_2 e^{-x} + \frac{1}{3} e^{2x} \end{pmatrix},$$

where c_1 and c_2 are arbitrary constants.

IV CONCLUSION

In this paper, we have presented a new method for solving system of higher order linear differential equations. This proposed method works for any vector forcing function of given system of HLDEs. The proposed method is developed using the basic concept of inverse of matrix and variation of parameters. Couple of examples (systems with constants coefficients and variable coefficients) are presented using proposed method to show the efficiency.

REFERENCES

[1] D. Sengupta: Resolution of the identity of the operator associated with a system of second order differential equations, J. Math. Comput. Sci. 5, No. 1, 56-71, 2015.

[2] D. Sengupta: On the expansion problem of a function associated with a system of second order differential equations, J. Math. Comput. Sci. 3, No. 6, 1565-1585, 2013.

[3] D. Sengupta: Asymptotic expressions for the eigenvalues and eigenvectors of a system of second order differential equations with a turning point (Extension II), International Journal of Pure and Applied Mathematics, Volume 78 No. 1, 85-95, 2012.

[4] S. Suksern, S. Moyo, S.V. Meleshko: Application of group analysis to classification of systems of three second-order ordinary differential equations, John Wiley \& Sons, Ltd. DOI: 10.1002/mma.3430, 2015.

[5] Vakulenko, S., Grigoriev, D. and Weber, A.: Reduction methods and chaos for quadratic systems of differential equations. Studies in Applied Mathematics, doi: 10.1111/sapm. 12083, 2015.

[6] Mukesh Grover: A New Technique to Solve Higher Order Ordinary Differential equations. IJCA Proceedings on National Workshop-Cum-Conference on Recent Trends in Mathematics and Computing 2011 RTMC, May 2012.

[7] R. Ben Taher and M. Rachidi: Linear matrix differential equations of higher-order and applications. Electronic Journal of Differential Equations, 2008(95), 1-12, 2008.

[8] G. I. Kalogeropoulos, A. D. Karageorgos, and A. A. Pantelous: Higher-order linear matrix descriptor differential equations of Apostol-Kolodner type. Electronic Journal of Differential Equations, 2009(25), 1-13, 2009.

[9] S. Abramov: EG - eliminations. Journal of Difference Equations and Applications, 5, 393-433, 1999.

[10] Carole EL BACHA: Methodes Algebriques pour la Resolution dEquations Differentielles Matricicelles d'Ordre Arbitraire. PhD thesis, LMC-IMAG, 2011.

[11] E. A. Coddington and R. Carlson: Linear ordinary differential equations. Society for Industrial and Applied Mathematics (SIAM), 1997.

[12] Apostol, Tom M.: Calculus, volume II. John Wiley \& Sons (Asia) Pte. Ltd., New York-Chichester-Brisbane-Toronto-Singapore (2002).

[13] Henry J. Ricardo: A Modern Introduction to Differential Equations, Second Edition, Academic Press, March, (2009).

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